

Covariant Theory of Gravitation in the Spacetime with Finsler Structure

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Abstract

The theory of gravitation in the spacetime with Finsler structure is constructed. It is shown that the theory keeps general covariance. Such theory reduces to Einstein's general relativity when the Finsler structure is Riemannian. Therefore, this covariant theory of gravitation is an elegant realization of Einstein's thoughts on gravitation in the spacetime with Finsler structure.

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[†]This paper is in memory of Prof. S.-S. Chern, a great mathematician I admire.

1 Introduction

In Newtonian mechanics, the spacetime is treated as a three-dimensional Euclidean space and an independent one-dimensional time. To solve the contradiction between the Newtonian spacetime and the Maxwell's electromagnetic theory, Einstein proposed the Minkowskian spacetime in his special relativity. Furthermore, Einstein introduced four-dimensional Riemannian spacetime in his gravitational field theory. After that, large amount of astronomical observations have shown that our spacetime is curved. Hence, the history of physics demonstrates that the geometry of our spacetime should be decided by the astronomical observations and the physical experiments.

At present our fundamental field theories are invariant under time reversal. But our everyday feelings tell us that there do exist a time arrow by which our world, our life and everything are controlled. So the direction of time should play a role in our theory on spacetime structure. The recent difficulties encountered in solving dark energy problems also hint that the geometry of our spacetime may be not Riemannian but its generalized case — Finsler geometry.

In this paper, we try to construct a covariant field theory of gravitation in the spacetime with the Finsler structure. We hope that this theory will provide a new powerful platform for solving the problems appeared in modern cosmology.

The paper is organized as follows: The Finsler structure in four-dimensional spacetime is introduced, and by which the fundamental tensor and the Cartan tensor are induced in Section 2. In Section 3 we introduce the Chern connection, which is the elegant mathematical tools in Finsler geometry. The curvatures of Chern connection are discussed, and several kinds of Bianchi identities are given in Section 4. We introduce a new tensor $\mathcal{Z}_{ijk}(x, y)$ in Section 5, by which a curvature-like tensor $\mathcal{H}_{ijkl}(x, y)$ is introduced. The covariant field equations of gravitation are acquired in this section also. In the last section, we discuss the strong constraint given by the conservation of the energy-momentum tensor. The vertical covariant derivative of the tensor $\mathcal{Z}_{ijk}(x, y)$

is discussed also.

2 Finsler Structure in four-dimensional Spacetime

For the spacetime of physics we need four coordinates, the time t and three physical space coordinates x^1, x^2, x^3 . We put $t = x^0$, so that the four coordinates may be written x^i , where the indice i takes on the four values 0, 1, 2, 3. Mathematically suppose that four-dimensional spacetime can be treated as a C^∞ manifold, denoted by M . Denote by $T_x M$ the tangent space at $x \in M$, and by $TM := \cup_{x \in M} T_x M$ the tangent bundle of M . Each element of TM has the form (x, y) , where $x \in M$ and $y \in T_x M$. The natural projection $\pi : TM \rightarrow M$ is given by $\pi(x, y) := x$. The dual space of $T_x M$ is $T_x^* M$, called the cotangent space at x . The union $T^* M := \cup_{x \in M} T_x^* M$ is the cotangent bundle. Therefore, $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$ are, respectively, the induced coordinate bases for $T_x M$ and $T_x^* M$. The said x^i give rise to local coordinates (x^i, y^i) on TM through the mechanism[‡]

$$y = y^i \frac{\partial}{\partial x^i} . \quad (1)$$

The y^i are fiberwise global. So functions $F(x, y)$ that are defined on TM can be locally expressed as $F(x^0, x^1, x^2, x^3; y^0, y^1, y^2, y^3)$.

In Ref.[1] a Finsler structure of an n -dimensional C^∞ manifold is given from the mathematical point of view. Similarly a Finsler Structure function of the four-dimensional spacetime can be defined globally

$$F : TM \rightarrow (-\infty, +\infty) \quad (2)$$

with the following properties:

[‡]In this paper, the rules that govern our index gymnastics are as follows:

1. Vector indices are up, and covector indices are down.
2. Any repeated pair of indices – provided that one is up and the other is down – is automatically summed.
3. The raising and lowering of indices are carried out by the matrix g_{ij} defined by equation (3), and its matrix inverse g^{ij} .

1. Regularity: $F(x, y)$ is C^∞ on the entire slit tangent bundle $TM_o := TM \setminus \{0\}$.
2. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.
3. The components of fundamental tensor:

$$g_{ij}(x, y) := \left[\frac{1}{2} F^2(x, y) \right]_{y^i y^j} , \quad (3)$$

where

$$\left[\frac{1}{2} F^2(x, y) \right]_{y^i y^j} := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} . \quad (4)$$

In Finsler geometry, the y^i (or x^i) appear in downstairs indices usually denote the partial derivatives with respect to the y^i (or x^i). In the spacetime with Finsler structure, the components of fundamental tensor can be decomposed by introducing the tetrad matrix $e_i^a(x, y)$ in local coordinates

$$g_{ij}(x, y) = e_i^a(x, y) e_j^b(x, y) \eta_{ab} . \quad (5)$$

we adopt the same sign conventions as that used by Misner-Thorne-Wheeler's book[2]. The metric tensor of local Minkowskian spacetime η_{ab} is written as follows

$$\eta_{00} = -1 , \quad \eta_{11} = \eta_{22} = \eta_{33} = +1 , \quad \eta_{ab} = 0 \quad \text{for} \quad a \neq b , \quad (6)$$

which is *not* positive-definite.

Given a Finsler structure $F(x, y)$ on the tangent bundle of four-dimensional spacetime M , the pair (M, F) can be called a **Finsler spacetime**.

In mathematical books on Finsler geometry, the Finsler structure function F satisfies $F \geq 0$ and the fundamental matrix $g_{ij}(x, y)$ be positive-definite at every point of TM_o . But in the case of four-dimensional spacetime, the tangent space is a Minkowskian spacetime, therefore $F(x, y)$ may not satisfy $F(x, y) \geq 0$ and $g_{ij}(x, y)$ is not positive-definite either.

Now we need the mathematical content of the pulled-back bundle π^*TM or its dual π^*T^*M . Here we don't explain how to construct this bundle, for more detailed discussions on the pulled-back bundle π^*TM or its dual π^*T^*M , please read Chapter 2 in Ref.[1]. For ease of local computations, it is to our advantage to work on an affine parameter space, where spacetime coordinates are readily available. In this case, the preferred base manifold is the slit tangent bundle TM_o . A good number of geometrical objects are sections of the pulled-back bundle π^*TM or its dual π^*T^*M , or their tensor products. These sit over TM_o and not M . Local coordinates $\{x^i\}$ on M produce the basis sections $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$, respectively, for T_xM and T_x^*M . Now, over each point (x, y) on the manifold TM_o , the fiber of π^*TM is the vector space T_xM while that of π^*T^*M is the covector space T_x^*M . Thus, the $\frac{\partial}{\partial x^i}$ and dx^i give rise to sections of the pulled-back bundles, in a rather simple-minded way. In the pulled-back bundles these sections are defined locally in x and globally in y . This global nature in y is automatic because once x is fixed, these sections do not change as we vary y .

Hence a distinguished section ℓ of π^*TM can be defined by

$$\ell = \ell_{(x,y)} := \frac{y}{F} = \frac{y^i}{F(x,y)} \frac{\partial}{\partial x^i} =: \ell^i \frac{\partial}{\partial x^i} . \quad (7)$$

Its natural dual is the Hilbert form ω , which is a section of π^*T^*M . We have

$$\omega = \omega_{(x,y)} := F_{y^i}(x,y) dx^i = F_{y^i} dx^i . \quad (8)$$

The definition (7) indicates that the components ℓ^i of the distinguished section ℓ satisfy $\ell^i = \frac{y^i}{F}$. According to Euler's theorem in Ref.[1], it is obvious that

$$\ell_i := g_{ij}(x,y) \ell^j = F_{y^i}(x,y) . \quad (9)$$

Thus the Hilbert form ω is expressible as $\omega = \ell_i dx^i$. Both ℓ and ω are globally defined on the manifold TM_o . The asserted duality means that

$$\omega(\ell) = \frac{y^i}{F} F_{y^i} = 1 , \quad (10)$$

which is a consequence of Euler's theorem too.

The pulled-back vector bundle π^*TM admits a natural Riemannian metric

$$g = g_{ij}dx^i \otimes dx^j , \quad (11)$$

where the components of g is defined by equation (3), obviously $g_{ij} = FF_{y^i y^j} + F_{y^i} F_{y^j}$ and $g_{ij} = g_{ji}$. This is the fundamental tensor, which determines the basic properties of the Finsler spacetime. It is a symmetric section of $\pi^*T^*M \otimes \pi^*T^*M$. Likewise, another important tensor in the Finsler spacetime is the Cartan tensor

$$A = A_{ijk}dx^i \otimes dx^j \otimes dx^k , \quad (12)$$

where the components is given by

$$A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4} [F^2]_{y^i y^j y^k} , \quad (13)$$

which is a totally symmetric section of $\pi^*T^*M \otimes \pi^*T^*M \otimes \pi^*T^*M$. Mathematically the object

$$C_{ijk} := \frac{A_{ijk}}{F} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} \quad (14)$$

is called the Cartan tensor in the geometric literature at large.

3 Chern Connection and Covariant Derivatives

The components g_{ij} of the fundamental tensor defined in equation (3) are functions on TM_o , and are invariant under the positive rescaling in y . We use them to define the formal Christoffel symbols of the second kind

$$\gamma^i_{jk} := \frac{g^{im}}{2} \left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m} \right) , \quad (15)$$

where g^{ij} is the matrix inverse of g_{ij} , and also the quantities

$$N^i_j := \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_{rs} y^r y^s . \quad (16)$$

The above quantities N^i_j can be reexpressed as follows

$$\frac{N^i_j}{F} := \gamma^i_{jk} \ell^k - A^i_{jk} \gamma^k_{rs} \ell^r \ell^s , \quad (17)$$

which is invariant under the positive rescaling $y \mapsto \lambda y$.

Let $[x^i = x^i(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3), i = 0, 1, 2, 3]$ be a change of coordinates on spacetime. Correspondingly, the chain rule gives

$$y^i = \frac{\partial x^i}{\partial \tilde{x}^j} \tilde{y}^j . \quad (18)$$

The tangent bundle of the manifold TM has a local coordinate basis that consists of the $\frac{\partial}{\partial x^i}$ and the $\frac{\partial}{\partial y^i}$. However, under the transformation on TM induced by a coordinate change $x \rightarrow \tilde{x}$, the vector $\frac{\partial}{\partial x^i}$ transforms in a complicated manner as follows:

$$\frac{\partial}{\partial \tilde{x}^j} = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial \tilde{x}^j \partial \tilde{x}^k} \tilde{y}^k \frac{\partial}{\partial y^i} . \quad (19)$$

On the other hand, the vector $\frac{\partial}{\partial y^i}$ transforms simply

$$\frac{\partial}{\partial \tilde{y}^j} = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial y^i} . \quad (20)$$

The cotangent bundle of the manifold T^*M has a local coordinate basis $\{dx^i, dy^i\}$. Here, under the said coordinate change, the dx^i behave simply

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j \quad (21)$$

while the dy^i transform complicatedly

$$d\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dy^j + \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} y^k dx^j . \quad (22)$$

To avoid the complexity in the transformation equations (19) and (22), furthermore, to obtain the coordinate bases that transform as tensor under the said coordinate change, Ref.[1] introduces two new symbols $\frac{\delta}{\delta x^i}$ and $\frac{\delta y^i}{F}$ to replace $\frac{\partial}{\partial x^i}$ and dy^i respectively. The $\frac{\delta}{\delta x^i}$ are defined by

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} , \quad (23)$$

and the $\frac{\delta y^i}{F}$ are given by

$$\frac{\delta y^i}{F} := \frac{1}{F} \left(dy^i + N^i_j dx^j \right) , \quad (24)$$

which is invariant under positive rescaling in y . Note that

$$\begin{array}{ccc} \frac{\delta}{\delta x^i} & \xrightarrow{\text{natural dual}} & dx^i, \\ F \frac{\partial}{\partial y^i} & \xrightarrow{\text{natural dual}} & \frac{\delta y^i}{F}. \end{array}$$

Therefore, we just introduce two new natural bases that are dual to each other: 1, the bases $\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i}\}$ for the tangent bundle of TM_o ; 2, the bases $\{dx^i, \frac{\delta y^i}{F}\}$ for the cotangent bundle of TM_o . The Ref.[1] indicates that the horizontal subspace spanned by the $\frac{\delta}{\delta x^i}$ is orthogonal to the vertical subspace spanned by the $F \frac{\partial}{\partial y^i}$.

The Chern connection is a linear connection that acts on the pulled-back vector bundle π^*TM , sitting over the manifold TM_o . It is *not* a connection on the bundle TM over M . Nevertheless, it serves Finsler geometry in a manner that parallels what the Levi-Civita connection (Christoffel symbol) does for Riemannian geometry. Here we cite Chern theorem on Chern connection in Ref.[1] in the following.

Chern Theorem: Let (M, F) be a Finsler manifold. The pulled-back bundle π^*TM admits a unique linear connection, called the Chern connection. Its connection forms are characterized by the structural equations:

1. **Torsion freeness:**

$$d(dx^i) - dx^j \wedge \omega_j^i = - dx^j \wedge \omega_j^i = 0. \quad (25)$$

2. **Almost g -compatibility:**

$$dg_{ij} - g_{kj} \omega_i^k - g_{ik} \omega_j^k = 2 A_{ijm} \frac{\delta y^m}{F}. \quad (26)$$

In fact, The torsion freeness is equivalent to the absence of dy^k terms in ω_j^i , that is to say

$$\omega_j^i = \Gamma_{jk}^i dx^k, \quad (27)$$

together with the symmetry

$$\Gamma_{jk}^i = \Gamma_{kj}^i. \quad (28)$$

Furthermore, almost metric-compatibility implies that

$$\Gamma^i_{jk} = \gamma^i_{jk} - g^{il} \left(A_{ljm} \frac{N^m_k}{F} + A_{klm} \frac{N^m_j}{F} - A_{jkm} \frac{N^m_l}{F} \right). \quad (29)$$

Equivalently,

$$\Gamma^i_{jk} = \frac{g^{il}}{2} \left(\frac{\delta g_{lk}}{\delta x^j} + \frac{\delta g_{jl}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^l} \right), \quad (30)$$

where the operators $\frac{\delta}{\delta x^i}$ have been defined by equation (23).

Using the Chern connection, the covariant derivatives of the tensors that are the sections of the pulled-back bundle π^*TM or its dual π^*T^*M , or their tensor product can be calculated. For instance, let $T := T^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ be an arbitrary smooth $(1,1)$ -type tensor, which sits on the manifold TM_o . Its covariant differential is

$$\nabla T := (\nabla T)^i_j \frac{\partial}{\partial x^i} \otimes dx^j, \quad (31)$$

where $(\nabla T)^i_j$ is

$$(\nabla T)^i_j := dT^i_j - T^i_k \omega^k_j + T^k_j \omega^i_k. \quad (32)$$

The components $(\nabla T)^i_j$ are 1-forms on TM_o . They are therefore be expanded in terms of the natural basis $\{dx^i\}$ and $\{\frac{\delta y^i}{F}\}$, set

$$(\nabla T)^i_j = T^i_{j|k} dx^k + T^i_{j;k} \frac{\delta y^k}{F}, \quad (33)$$

where $T^i_{j|k}$ is the horizontal covariant derivative of $(\nabla T)^i_j$ and $T^i_{j;k}$ is the vertical covariant derivative of $(\nabla T)^i_j$ respectively. In order to obtain formulas for the coefficients, we evaluate equation (33) on each individual member of the dual basis $\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i}\}$. We also use the fact that the Chern connection forms for the natural basis have no $\frac{\delta y^k}{F}$ terms, and are given by equation (27). Therefore the results are

$$T^i_{j|k} = \left(\nabla_{\frac{\delta}{\delta x^k}} T \right)^i_j = \frac{\delta T^i_j}{\delta x^k} + T^l_j \Gamma^i_{lk} - T^i_l \Gamma^l_{jk}, \quad (34)$$

$$T^i_{j;k} = \left(\nabla_{F \frac{\partial}{\partial y^k}} T \right)^i_j = F \frac{\partial T^i_j}{\partial y^k}. \quad (35)$$

The treatment for tensor fields of higher or lower rank is similar. Here we list the covariant derivatives of several important tensors. First Chern theorem says that the

Chern connection is almost g -compatible, namely

$$(\nabla g)_{ij} = dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 2 A_{ijl} \frac{\delta y^l}{F} . \quad (36)$$

This shows that the covariant derivatives of fundamental tensor are

$$g_{ij|l} = 0 , \quad (37)$$

$$g_{ij;l} = 2 A_{ijl} . \quad (38)$$

The obvious equations $(g^{ij}g_{jk})_{|l} = 0$ and $(g^{ij}g_{jk})_{;l} = 0$ yield

$$g^{ij}_{|l} = 0 , \quad g^{ij}_{;l} = -2A^{ij}_l . \quad (39)$$

Secondly, the covariant derivatives of the distinguished ℓ are

$$\ell^i_{|j} = 0 , \quad (40)$$

$$\ell^i_{;j} = \delta^i_j - \ell^i \ell_j . \quad (41)$$

These, together with (37) and (38), can then be used to deduce that

$$\ell_{i|j} = 0 , \quad (42)$$

$$\ell_{i;j} = g_{ij} - \ell_i \ell_j . \quad (43)$$

Those covariant derivatives will be used in the process of constructing the field equations of gravitation.

4 Curvature and Bianchi Identities

The curvature 2-forms of the Chern connection are

$$\Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i . \quad (44)$$

Since the Ω_j^i are 2-forms on the manifold TM_o , Chern proved that they can be expanded as

$$\Omega_j^i = \frac{1}{2} R_j^i{}_{kl} dx^k \wedge dx^l + P_j^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} . \quad (45)$$

The objects R , P are respectively the hh -, hv -curvature tensors of the Chern connection. The wedge product $dx^k \wedge dx^l$ in above equation demonstrates that

$$R_j^i{}_{kl} = - R_j^i{}_{lk} . \quad (46)$$

The first Bianchi identities deduced from the torsion freeness of the Chern connection uncovers a symmetry on $P_j^i{}_{kl}$

$$P_j^i{}_{kl} = P_k^i{}_{jl} \quad (47)$$

and the first Bianchi identity for $R_j^i{}_{kl}$

$$R_j^i{}_{kl} + R_k^i{}_{lj} + R_l^i{}_{jk} = 0 . \quad (48)$$

In natural coordinates, formulas for $R_j^i{}_{kl}$ and $P_j^i{}_{kl}$ are expressed in terms of the Chern connection Γ_{jk}^i as follows

$$R_j^i{}_{kl} = \frac{\delta \Gamma_{jl}^i}{\delta x^k} - \frac{\delta \Gamma_{jk}^i}{\delta x^l} + \Gamma_{hk}^i \Gamma_{jl}^h - \Gamma_{hl}^i \Gamma_{jk}^h , \quad (49)$$

and

$$P_j^i{}_{kl} = - F \frac{\partial \Gamma_{jk}^i}{\partial y^l} . \quad (50)$$

Note that equations (28) and (50) imply (47).

The Chern connection is almost metric-compatible. Using this property, some Bianchi identities are found. After exterior differentiation on equation (26) and some manipulations, we get

$$\begin{aligned} & \Omega_{ij} + \Omega_{ji} \\ &= \frac{1}{2} (R_{ijkl} + R_{jikl}) dx^k \wedge dx^l + (P_{ijkl} + P_{jikl}) dx^k \wedge \frac{\delta y^l}{F} \\ &= - (A_{iju} R_{kl}^u) dx^k \wedge dx^l - 2(A_{iju} P_{kl}^u + A_{ijl|k}) dx^k \wedge \frac{\delta y^l}{F} \\ & \quad + 2(A_{ijk;l} - A_{ijk} \ell_l) \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F} . \end{aligned} \quad (51)$$

Here, we have introduced the abbreviations

$$R_{kl}^i := \ell^j R_j^i{}_{kl} . \quad (52)$$

$$P_{kl}^i := \ell^j P_j^i{}_{kl} . \quad (53)$$

There are three identities that one can obtain from above equation (51). We carry them out systematically. The coefficients of the $dx^k \wedge dx^l$ terms in (51) tell us that

$$R_{ijkl} + R_{jikl} = -2 A_{iju} R_{kl}^u. \quad (54)$$

The coefficients of the $dx^k \wedge \frac{\delta y^l}{F}$ terms in (51) tell us that

$$P_{ijkl} + P_{jikl} = -2 A_{iju} P_{kl}^u - 2 A_{ijl|k}. \quad (55)$$

Apply (55) three times to the combination

$$(P_{ijkl} + P_{jikl}) - (P_{jkil} + P_{kji l}) + (P_{kijl} + P_{ikjl}). \quad (56)$$

Through some operation, the result takes the form

$$P_{jikl} = - (A_{ijl|k} - A_{jkl|i} + A_{kil|j}) - (A_{iju} P_{kl}^u - A_{jku} P_{il}^u + A_{kiu} P_{jl}^u). \quad (57)$$

Contract this equation with ℓ^j , respectively, adopt (40) and the facts $P_{jk}^i \ell^k = 0$, $\ell^i A_{ijk} = 0$, we then reduces the contraction to the important statement

$$P_{ikl} := \ell^j P_{jikl} = -\dot{A}_{ikl}. \quad (58)$$

Here,

$$\dot{A}_{ikl} := A_{ikl|m} \ell^m. \quad (59)$$

Then (57) and (58) together lead to the constitutive relation for P_{jikl}

$$P_{jikl} = - (A_{ijl|k} - A_{jkl|i} + A_{kil|j}) + (A_{ij}^u \dot{A}_{ukl} - A_{jk}^u \dot{A}_{uil} + A_{ki}^u \dot{A}_{ujl}). \quad (60)$$

Formula (58) can be used to reexpress equation (55) as

$$A_{ijl|k} = A_{ij}^u \dot{A}_{ukl} - \frac{1}{2} (P_{ijkl} + P_{jikl}). \quad (61)$$

Finally, the coefficients of the $\frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}$ terms in (51) gives

$$A_{ijk;l} - A_{ijl;k} = A_{ijk} \ell_l - A_{ijl} \ell_k. \quad (62)$$

So we have discussed all formulas taken from (51).

Exterior differentiation of (44) gives the second Bianchi identity

$$d\Omega_j^i - \omega_j^k \wedge \Omega_k^i + \omega_k^i \wedge \Omega_j^k = 0. \quad (63)$$

With (27) and (45), one can expand the above equation as follows

$$\begin{aligned} 0 &= \frac{1}{2} \left(R_j^i{}_{kl|t} - P_j^i{}_{ku} R_{lt}^u \right) dx^k \wedge dx^l \wedge dx^t \\ &+ \frac{1}{2} \left(R_j^i{}_{kl;t} - 2 P_j^i{}_{kt|l} + 2 P_j^i{}_{ku} \dot{A}_{lt}^u \right) dx^k \wedge dx^l \wedge \frac{\delta y^t}{F} \\ &+ \left(P_j^i{}_{kl;t} - P_j^i{}_{kl} \ell_t \right) dx^k \wedge \frac{\delta y^l}{F} \wedge \frac{\delta y^t}{F}. \end{aligned}$$

The above equation is equivalent to the following three identities:

$$R_j^i{}_{kl|t} + R_j^i{}_{lt|k} + R_j^i{}_{tk|l} = P_j^i{}_{ku} R_{lt}^u + P_j^i{}_{lu} R_{tk}^u + P_j^i{}_{tu} R_{kl}^u, \quad (64)$$

$$R_j^i{}_{kl;t} = P_j^i{}_{kt|l} - P_j^i{}_{lt|k} - \left(P_j^i{}_{ku} \dot{A}_{lt}^u - P_j^i{}_{lu} \dot{A}_{kt}^u \right), \quad (65)$$

$$P_j^i{}_{kl;t} - P_j^i{}_{kt;l} = P_j^i{}_{kl} \ell_t - P_j^i{}_{kt} \ell_l. \quad (66)$$

Making use of (40), Contracting (64) with ℓ^j , one can obtain

$$R^i{}_{kl|t} + R^i{}_{lt|k} + R^i{}_{tk|l} = - \dot{A}^i{}_{ku} R_{lt}^u - \dot{A}^i{}_{lu} R_{tk}^u - \dot{A}^i{}_{tu} R_{kl}^u. \quad (67)$$

In general relativity, Einstein used the second Bianchi identity in Riemannian geometry to deduce the important Einstein tensor, the contraction of whose covariant derivative is vanished. As we will show, the second Bianchi identity in Finsler geometry also plays an important role in building the elegant field equations of gravitation.

5 The Field Equations of Gravitation in Finsler Spacetime

We find that a tensor $\mathcal{Z}_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k$ is necessary in constructing the covariant field equations of gravitation. The tensor $\mathcal{Z}_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k$ is described by

$$\mathcal{Z}_{ijl|k} = \mathcal{Z}_{iju} \dot{A}_{kl}^u - P_{ijkl}. \quad (68)$$

By comparing (61) with (68), we obtain a simple relationship

$$A_{ijk} = \frac{1}{2} (\mathcal{Z}_{ijk} + \mathcal{Z}_{jik}) . \quad (69)$$

Above equation shows that $\mathcal{Z}_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ is closely related with the Cartan tensor $A_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$. From the definition of Chern connection (29), one easily obtains

$$\Gamma_{ijk} + \Gamma_{jik} = \frac{\partial g_{ij}}{\partial x^k} - 2A_{ijl} \frac{N_k^l}{F} . \quad (70)$$

Combining (68), (69) and (70), we conclude that

$$\Gamma_{ijk} + \mathcal{Z}_{jil} \frac{N_k^l}{F} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) . \quad (71)$$

Namely

$$\Gamma_{jk}^i + \mathcal{Z}_j^i \frac{N_k^l}{F} = \gamma_{jk}^i . \quad (72)$$

The formula (72) demonstrates that the tensor $\mathcal{Z}_j^i(x, y)$ vanishes when the Finsler structure is Riemannian.

We introduce a curvature-like tensor \mathcal{H}_{ijkl} as follows

$$\mathcal{H}_{ijkl} := R_{ijkl} + \mathcal{Z}_{iju} R_{kl}^u , \quad (73)$$

which is a combination of hh -curvature tensor R_{ijkl} and an additional revised-curvature tensor $\mathcal{Z}_{iju} R_{kl}^u$. The definition (73) and formula (54) tell us that

$$\mathcal{H}_{ijkl} = - \mathcal{H}_{jikl} . \quad (74)$$

The definition (73) and formula (46) tell us that

$$\mathcal{H}_{ijkl} = - \mathcal{H}_{ijlk} . \quad (75)$$

In the curvature-like tensor \mathcal{H}_{ijkl} , the second Bianchi identity on R_{ijkl} has been given, namely equation (64). Now we try to find a similar identity on $\mathcal{Z}_j^i R_{kl}^u$. First

$$\left(\mathcal{Z}_j^i R_{kl}^u \right)_{|t} = \mathcal{Z}_j^i{}_{u|t} R_{kl}^u + \mathcal{Z}_j^i R_{kl|t}^u . \quad (76)$$

Apply (76) three times to the following combination

$$\left(\mathcal{Z}_j^i{}^u R^u{}_{kl}\right)_{|t} + \left(\mathcal{Z}_j^i{}^u R^u{}_{lt}\right)_{|k} + \left(\mathcal{Z}_j^i{}^u R^u{}_{tk}\right)_{|l} . \quad (77)$$

With the help of (67), (68) and (76), we can get the result of expression (77) as follows

$$\begin{aligned} & \left(\mathcal{Z}_j^i{}^u R^u{}_{kl}\right)_{|t} + \left(\mathcal{Z}_j^i{}^u R^u{}_{lt}\right)_{|k} + \left(\mathcal{Z}_j^i{}^u R^u{}_{tk}\right)_{|l} \\ = & - P_j^i{}_{ku} R^u{}_{lt} - P_j^i{}_{lu} R^u{}_{tk} - P_j^i{}_{tu} R^u{}_{kl} . \end{aligned} \quad (78)$$

Therefore, combining (64) with (78), we have an important identity

$$\mathcal{H}_j^i{}_{kl|t} + \mathcal{H}_j^i{}_{lt|k} + \mathcal{H}_j^i{}_{tk|l} = 0 . \quad (79)$$

We shall be particularly concerned with the contracted form of (79). Recalling that the horizontal covariant derivatives of g^{jl} vanish, we find on contraction of j with l that

$$\mathcal{H}^i{}_{k|t} - \mathcal{H}^i{}_{t|k} + \mathcal{H}^{ji}{}_{tk|j} = 0 , \quad (80)$$

where we have adopted the definition[§]

$$\mathcal{H}_{kj} := g^{il} \mathcal{H}_{ikjl} = \mathcal{H}^i{}_{kji} . \quad (82)$$

Contracting equation (80) again gives

$$\mathcal{H}_{|t} - \mathcal{H}^i{}_{t|i} - \mathcal{H}^j{}_{t|j} = 0 , \quad (83)$$

or

$$\left(\mathcal{H}^j{}_t - \frac{1}{2} \delta^j_t \mathcal{H}\right)_{|j} = 0 , \quad (84)$$

where the scalar \mathcal{H} is

$$\mathcal{H} := g^{ij} \mathcal{H}_{ij} = \mathcal{H}^i{}_i . \quad (85)$$

[§]We strongly feel that \mathcal{H}_{kj} satisfy $\mathcal{H}_{kj} = \mathcal{H}_{jk}$, but till now we cannot prove it. But we believe that geometers can find it. If not, we have to redefine \mathcal{H}_{ijkl} as follows

$$\mathcal{H}_{ijkl} := \frac{1}{2} \left(R_{ijkl} + R_{kl ij} + \mathcal{Z}_{iju} R^u{}_{kl} + \mathcal{Z}_{klu} R^u{}_{ij} \right) . \quad (81)$$

This re-definition will not change our field equations. But the theory will be slightly ugly.

An equivalent but more familiar form for (84) is

$$\left(\mathcal{H}^{jt} - \frac{1}{2} g^{jt} \mathcal{H} \right)_{|j} = 0 . \quad (86)$$

In general relativity, the energy-momentum tensor T_{ij} is conserved to make sure the conservation of energy and momentum [3]. Similarly, we introduce the energy-momentum tensor $\mathcal{T}_{ij}(x, y)dx^i \otimes dx^j$ in the manifold π^*TM . The covariant derivatives of the energy-momentum tensor are

$$(\nabla \mathcal{T})^{ij} = \mathcal{T}^{ij}_{|k} dx^k + \mathcal{T}^{ij}_{;k} \frac{\delta y^k}{F} , \quad (87)$$

where $\mathcal{T}^{ij}_{|k}$ are the horizontal covariant derivatives of $\mathcal{T}_{ij}(x, y)$ and $\mathcal{T}^{ij}_{;k}$ are the vertical covariant derivatives of $\mathcal{T}_{ij}(x, y)$. When the Finsler structure is Riemannian, $\mathcal{T}^{ij}_{;k}$ vanish, and the horizontal covariant derivatives $\mathcal{T}^{ij}_{|k}$ become the covariant derivatives of the energy-momentum tensor. Therefore, from the physical point of view, the energy and momentum conservation can be kept when the horizontal covariant derivatives $\mathcal{T}^{ij}_{|k}$ of the energy-momentum tensor satisfy

$$\mathcal{T}^{ij}_{|i} = 0 . \quad (88)$$

Frankly speaking, we still do not understand the physical meaning of the vertical covariant derivatives of the energy-momentum tensor.

With the energy-momentum tensor $\mathcal{T}_{ij}(x, y)$ sitting over the Finsler spacetime, when its horizontal derivatives satisfy (88), we propose the covariant field equations of gravitation as follows

$$\mathcal{H}^{ij} - \frac{1}{2} g^{ij} \mathcal{H} = 8\pi G \mathcal{T}^{ij} , \quad (89)$$

where G is the Newtonian constant. Obviously, (86) makes sure that the energy-momentum tensor in (89) satisfies (88). Therefore, the field equations (89) do reserve the energy and momentum conservation.

Obviously, for the empty spacetime, the energy-momentum tensor in (89) disappears, the field equations of gravitation reduce to

$$\mathcal{H} = 0 . \quad (90)$$

When the Finsler structure is Riemannian, the tensor $\mathcal{Z}_{iju}R^u_{kl}$ vanishes and the curvature-like tensor \mathcal{H}_{ijkl} becomes the Riemannian curvature tensor of Riemannian spacetime, our field equations becomes Einstein's field equations exactly. Therefore, our covariant field equations of gravitation in the Finsler spacetime are the natural result of Einstein's thoughts on gravitation.

6 More Discussions

The equation (87) demonstrates that the energy-momentum conservation in Finsler spacetime can be divided into two kinds. First, the weak energy-momentum conservation, used in last section. In this kind of energy-momentum conservation, we require that $\mathcal{T}^{ij}_{|i} = 0$, the field equations of gravitation has been discussed. Secondly, the strong energy-momentum conservation, we call, that is, both $\mathcal{T}^{ij}_{|i}$ and $\mathcal{T}^{ij}_{;i}$ vanish. The constraint condition

$$\mathcal{T}^{ij}_{;i} = 0 \quad (91)$$

sets a serious constraint on the vertical covariant derivatives of tensor \mathcal{Z}_{ijk} . If conservation (91) is needed, the field equations (89) tell us that

$$\left(\mathcal{H}^{ij} - \frac{1}{2} g^{ij} \mathcal{H} \right)_{;i} = 0. \quad (92)$$

But we cannot draw a concise equation on $\mathcal{Z}_{ijk;l}$ from (92). We find that the constraint equation of $\mathcal{Z}_{ijk;l}$ given by (92) is so complicated that no valuable information can be obtained. Maybe, further studies will demonstrate that (68) and (92) are exclusive, then the strong energy-momentum conservation cannot be required in the spacetime with the Finsler structure.

In a conclusion, we use the Chern connection and the curvatures given by the Chern connection to set up the field equations of gravitation in the Finsler spacetime. After introducing the tensor \mathcal{Z}_{ijk} and its related curvature-like tensor \mathcal{H}^i_{jkl} , we obtain the field equations of gravitation, formally like Einstein's field equations. We show that our

field equations exactly reduces to Einstein's field equations when the Finsler structure is Riemannian.

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